Kinetic Equations

Text of the Exercises

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Exercise 1

Consider a family of probability measures $F \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$. We say that *the family is tight* if for any $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^d$ such that $\mu(K) > 1 - \varepsilon$ for any $\mu \in F$.

The following Theorem holds true (for a proof, see Theorem 5.1 in *Convergence of Probability Measures (Second Edition)* by P. Billingsley).

Theorem (Prohorov's Theorem). *Consider a sequence of probability measures $\{\mu_k\}_{k\in\mathbb{N}} \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$ which is tight. Then there exists a subsequence*[1] *$\{\mu_{k_l}\}_{l\in\mathbb{N}}$ and a probability measure $\mu \in \mathcal{P}\left(\mathbb{R}^d\right)$ such that $\mu_{k_l} \rightharpoonup \mu$ if $l \to +\infty$.*

Use Prohorov's Theorem (without proving it) to prove that $\left(\mathcal{P}_1\left(\mathbb{R}^d\right), \mathcal{W}_1\right)$ is a complete metric space.

Hint: Consider a Cauchy sequence for $\mathcal{W}_1$, show that it is tight. Using the Theorem deduce the existence of a weak limit and prove that the convergence holds also with the metric $\mathcal{W}_1$.

Exercise 2

Prove the second part of Dobrushin's Theorem, i.e., let $M = \{\mu_t |\ t \in [0,T]\} \in \mathfrak{M}_T^+(\mu_0)$ a solution to

$$\begin{cases} \partial_t \mu_t(\psi) = \mu_t\left(v \cdot \nabla_x \psi + E_{\mu_t} \cdot \nabla_v \psi\right), & t \in [0,T],\ \psi \in C_c^\infty\left(\mathbb{R}^3 \times \mathbb{R}^3\right), \\ \mu_t(\psi)|_{t=0} = \mu_0(\psi), & \psi \in C_c^\infty\left(\mathbb{R}^3 \times \mathbb{R}^3\right), \end{cases} \tag{1}$$

with

$$E_\mu(x) = \int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U\left(x - x'\right) d\mu\left(x', v'\right), \qquad U \in C_b^2\left(\mathbb{R}^3\right), \tag{2}$$

and with μ_0 absolutely continuous with respect to $\mathcal{L}^6$, i.e. $d\mu_0(x,v) = f_0(x,v)\,dxdv$. Prove that if $f_0 \in C^1\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$ then also μ_t is absolutely continuous with respect to $\mathcal{L}^6$ and moreover $d\mu_t(x,v) = f(t,x,v)\,dxdv$ with $f \in C^1\left([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3\right)$.

Hint: It can be convenient to use the fact that, once the solution exists, the Vlasov equation can be seen as a Liouville equation with a potential depending on the existing solution.

[1] Recall that $\{\mu_{k_l}\}_{l\in\mathbb{N}}$ is a subsequence of $\{\mu_k\}_{k\in\mathbb{N}}$ if the sequence $\{k_l\}_{l\in\mathbb{N}}$ is a sequence of natural numbers such that $k_{l+1} > k_l$ for any $l \in \mathbb{N}$.

Exercise 3

Let $f_0 \in L^1 \cap L^\infty\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$; consider the following initial value problem:

$$\left\{\begin{array}{ll} \partial_t f + v \cdot \nabla_x f = 0, & \text{in } \mathcal{D}'\left([0,+\infty) \times \mathbb{R}^3 \times \mathbb{R}^3\right), \\ f|_{t=0} = f_0, & \text{in } \mathcal{D}'\left(\mathbb{R}^3 \times \mathbb{R}^3\right), \end{array}\right. \tag{3}$$

where we also assume as usual that the map $t \mapsto \langle f(t,\cdot,\cdot), \varphi\rangle$ is continous in t for any $\varphi \in C_c^\infty\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$.

(i) Prove that there exists a unique solution to (3) and show its explicit form.

(ii) Use the explicit form to prove that

$$\|f(t,\cdot,\cdot)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} = \|f_0\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}, \qquad \forall t \in [0,+\infty),\ p \in [1,+\infty]. \tag{4}$$

(iii) Use the explicit form to prove the following dispersion relation:

$$\|f(t,\cdot,\cdot)\|_{L^\infty_x(\mathbb{R}^3;L^1_v(\mathbb{R}^3))} \leqslant \frac{1}{|t|^3}\|f_0\|_{L^1_x(\mathbb{R}^3;L^\infty_v(\mathbb{R}^3))}, \qquad \forall t \in (0,+\infty). \tag{5}$$